Unsteady Hypersonic Flow over Delta Wings with Detached Shock Waves

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The problem of pitching oscillating slender delta wings with detached shock waves in hypersonic flow is studied using Messiter's thin shock-layer theory. The amplitude of oscillation is assumed small and a perturbation method is employed. Closed-form simple formulas are obtained for the unsteady pressure field and for the aerodynamic derivatives of the delta wings which are valid for general frequencies. It is found that within the thin shock-layer approximation, the slender delta wings with detached shock waves pitching in hypersonic stream are always stable dynamically. An accurate perturbation solution to Meissiter's functional-differential equation, which is required in calculating the steady and unsteady flowfields, is also obtained.

I. Introduction

THE problem of unsteady two-dimensional flow with attached shock waves has been well studied. In particular, the case of oscillating wedges with small amplitude in inviscid flow was recently studied by Appleton,[1] McIntosh,[2] and Hui,[3,4] among others. The case of large amplitude slow oscillation was tackled by Kuiken[5] for slender wedges and by Hui[6] for general wedges, whilst the effects of viscosity on the stability of slender wedges at high Mach numbers were treated by Orlik-Ruckemann[7] and by Hui and East.[8]

In contrast, little theoretical work has been done on the three-dimensional problems of an oscillating wing with attached or detached shock waves. Theoretical studies of problems of this type are evidently important in predicting the aerodynamic stability of a re-entry hypersonic/supersonic vehicle, such as the space shuttle, during its course of atmospheric flight.

One of the most powerful methods for attacking problems of unsteady flow with shock waves is the perturbation method in which the unsteady flow is regarded as a small perturbation of some steady reference flow. The success of the pertubation method and the accuracy of the resulting unsteady-flow solution depend critically on having an accurate steady reference flow solution from which to perturb. Thus the exact uniform steady wedge flow solution has been utilized in finding the oscillating wedge solution. On the other hand, for a delta wing there now exist the steady-flow solution of Hui[9,10] for the attached shock flow case and the corresponding steady-flow solution of Messiter[11] for the detached shock case. Perturbing Hui's solution, Liu and Hui[12] have recently developed an analytical theory for predicting the aerodynamic stability of a (pitching) oscillating delta wing with shock waves attached to the leading edges. It is uniformly valid for both hypersonic and supersonic flows.

The purpose of the present paper is to study the aerodynamic stability of an (pitching) oscillating slender delta wing in a hypersonic stream with shock wave detached from the leading edges but attached to the wing apex. This is done by extending Messiter's steady-flow theory to the unsteady case.

The problem considered by Messiter[11] is that of the steady hypersonic flow of gas past a slender flat delta wing symmetrically placed at a moderate angle of attack α. Assuming a nonviscous, nonheat-conducting perfect gas of constant specific heats, Messiter perturbed the Newtonian flow for small values of

$$\epsilon = \frac{\gamma - 1}{\gamma + 1} + \frac{2}{(\gamma + 1) M_\infty^2 \sin^2 \alpha} \tag{1}$$

where M_∞ is the freestream Mach number and γ the ratio of specific heats of the gas. He studied the interesting case for which the wing vertex angle and the Mach angle are of the same order of magnitude as $\epsilon \to 0$, and also characterized this case by the constancy of the similarity parameter.

$$\Omega = \frac{\bar{b}}{\bar{\ell} \epsilon^{1/2} \tan \alpha} \tag{2}$$

where $\bar{\ell}$ is the length and $\bar{b}$ the semi-span of the wing. Messiter pointed out that whether the shock waves are attached to the leading edges or are attached only to the apex of the wing depends on whether Ω is greater or less than 2. The latter case will be referred to as detached shock flow case for simplicity. By an ingenious method, Messiter obtained a first-order correction to the Newtonian approximation in the detached shock case and found it generally in good agreement with experimental data. Messiter's theory has since been extended by Hida[13] to accounting for the effects of wing thickness, and by Squire[13] to wings of diamond and caret sections.

In this paper the same delta wing configuration as in Messiter is considered to be oscillating (pitching) about an arbitrary axis of the wing, and the problem is to calculate the resulting unsteady aerodynamic forces and in particular the aerodynamic derivatives of the wing. In addition to the assumptions made by Messiter,[11] the amplitude of oscillation of the wing, λ_0, is assumed to be small. A double series expansion solution in ϵ and λ_0 is assumed and shown to lead to consistent equations and boundary conditions. It contains two correction terms to the Newtonian limiting solution. The first correction term takes care of the effects of small but finite ϵ, while the second correction term takes care of the small-amplitude oscillations of the wing. The special cases of steady flow and unsteady Newtonian flow can be recovered easily by double series expansion. The frequency of oscillation need not be assumed small and the present theory is valid for values of the reduced frequency parameter up to order unity. The formulation of the perturbation problem and solution for the unsteady pressure are given in Sec. II, whereas closed form formulas for the aerodynamic derivatives of the delta wing are given in Sec. III. Appendix A is devoted to a perturbation solution of the functional-differential equation, which is

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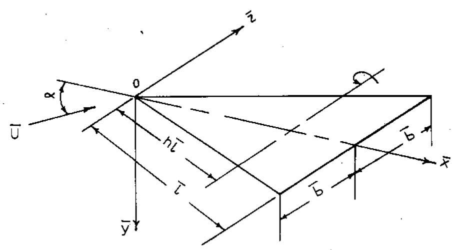

Fig. 1 Wing geometry and coordinate system.

required in calculating the steady flow and in calculating the stability derivatives.

II. The Perturbation Scheme and Solution

Consider a flat delta wing (Fig. 1) with triangular planform placed symmetrically in a hypersonic stream of gas. Let the mean angle of attack be α. A shock wave exists below the wing. Provided that α is not close to $\pi/2$, the shock is either attached only to the wing vertex or is attached to the leading edges, depending on the values of the parameter Ω. In this paper, we are concerned with the case when the shock is attached to the wing only at the vertex. A cartesian coordinate system $0\bar{x}\bar{y}\bar{z}$, fixed in space, is chosen such that the wing in its mean position lies in the plane $\bar{y}=0$. Let the wing make small pitching oscillations about an axis on the wing surface at a distance $h\bar{\ell}$ from the wing vertex. The wing surface will be given by (in what follows we consider only the real parts of all complex expressions.)

$$B(\bar{x},\bar{y},\bar{z},\bar{t}) \equiv \bar{y} - \lambda_0 e^{i\omega\bar{t}}(h\bar{\ell}-\bar{x}) = 0 \tag{3}$$

where ω is the circular frequency and $\bar{t}$ the time variable. The region of interest is the high-pressure region bounded by the wing surface, the shock wave, and the planes $\bar{z}=\pm b\bar{x}$, where $b=\bar{b}/\bar{\ell}$. The upper surface has very little contribution to aerodynamic forces in hypersonic flow and will be neglected entirely.

It should be pointed out that according to gas dynamics theory, the detached shock flow (which is the case studied in this paper) is governed by a system of conically elliptic equations. An interesting feature of Messiter's thin shock-layer approximation is that it changes the governing equations to conically hyperbolic type in the whole flowfield, thus making the boundary value problems for the flow calculation easier to solve.

At the same time such approximation and the consequent alteration of the essential characteristics of the governing equations create certain difficulties for the attached shock-flow case, as the boundary value problem for calculating the flow then appears to be overdetermined. However, a different approach to the attached shock case has been successfully developed by Hui.[9] It is worth noting that the advantageous aspect of Messiter's thin shock-layer formulation for the detached shock case is retained in the present unsteady flow calculations, making it possible to obtain closed-form solution for the unsteady pressure field.

Based on the assumptions stated in the introduction, the governing equations of continuity, momentum, and energy for the pressure $\bar{p}$, density $\bar{\rho}$, and the componets $\bar{u}$, $\bar{v}$, $\bar{w}$ of the velocity $\bar{q}$ may be written in the form

$$\bar{\rho}_{\bar{t}} + \nabla\cdot(\bar{\rho}q) = 0 \tag{4a}$$

$$q_{\bar{t}} + q\cdot\nabla q + \frac{1}{\rho}\nabla\bar{p} = 0 \tag{4b}$$

$$(\bar{p}/\bar{\rho}^{\gamma})_{\bar{t}} + q\cdot\nabla(\bar{p}/\bar{\rho}^{\gamma}) = 0 \tag{4c}$$

The boundary condition to be satisfied at wing surface is that the normal component of relative velocity vanishes; that is,

$$B_{\bar{t}} + q\cdot\nabla B = 0 \tag{5}$$

Using Eq. (3), this becomes

$$-\lambda_0 i\omega e^{i\omega\bar{t}}(h\bar{\ell}-\bar{x}) + \lambda_0 e^{i\omega\bar{t}}\bar{u} + \bar{v} = 0, \quad \text{at} \quad B=0 \tag{6}$$

The conditions to be satisfied at the shock surface are the Rankine-Hugoniot jump conditions:

$$[\bar{\rho}(S_{\bar{t}}+q\cdot\nabla S)] = 0 \tag{7a}$$

$$[\bar{\rho}(S_{\bar{t}}+q\cdot\nabla S)^2 + (\nabla S)^2\bar{p}] = 0 \tag{7b}$$

$$[\tfrac{1}{2}(S_{\bar{t}}+q\cdot\nabla S)^2 + (\nabla S)^2[\gamma/(\gamma-1)](\bar{p}/\bar{\rho})] = 0 \tag{7c}$$

$$[q\times\nabla S] = 0 \tag{7d}$$

where the square brackets denote the change in the enclosed quantity across the shock and S is the shock wave equation given by

$$S(\bar{x},\bar{y},\bar{z},\bar{t}) \equiv \bar{y} - \bar{y}_s(\bar{x},\bar{z},\bar{t}) = 0 \tag{8}$$

$\bar{y}_s(\bar{x},\bar{z},\bar{t})$ is the shock height, as yet unknown, which is to be found as a part of the solution. Parts a, b, and c of Eq. (7) are the usual conservation equations of mass, momentum, and specific enthalpy across a normal shock, the last is a vector equation (equivalent to two scalar equations) expressing continuity of tangential velocity. Equation system (7) is sufficient to allow calculation of values of the functions $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{p}$, $\bar{\rho}$, just behind the shock surface, in terms of the unknown shock shape $\bar{y}_s$.

The Newtonian approximation is reached as $\gamma\to1$ and $M_\infty\to\infty$ independently. Let $\bar{U}$ denote the velocity, $\bar{p}_\infty$ the pressure and $\bar{\rho}_\infty$ the density in the freestream, then in the Newtonian limit it is expected, throughout the region of interest, that

$$\bar{u}\to\bar{U}\cos\alpha, \qquad \bar{v}\to0, \qquad \bar{w}\to0$$
$$\bar{p}\to\bar{\rho}_\infty\bar{U}^2\sin^2\alpha, \qquad \bar{\rho}_\infty/\bar{\rho}\to\epsilon \tag{9}$$

As mentioned in the introduction, we consider the special limiting process for which Ω remains fixed and its value is less than 2, as $\epsilon\to0$. In this case the shock wave will be detached from the leading edges (but attached at the vertex) of the wing for

$$0\le\Omega\le2$$

The nondimensional variables are defined in such a manner that they remain fixed in the flowfield of interest regardless of the limits $\epsilon\to0$, $\lambda_0\to0$, independently; that is, let

$$x=\frac{\bar{x}}{\bar{\ell}}, \qquad y=\frac{\bar{y}-\bar{y}_b}{\bar{\ell}\bar{\epsilon}\tan\alpha}$$

$$z=\frac{\bar{z}}{\bar{\ell}\bar{\epsilon}^{1/2}\tan\alpha}, \qquad t=\frac{\bar{t}\bar{U}\cos\alpha}{\bar{\ell}} \tag{10}$$

where

$$\bar{y}_b=\lambda_0 e^{i\omega\bar{t}}(h\bar{\ell}-\bar{x})$$

The asymptotic expansion solution therefore should be expressed in terms of these variables. The following expansions of the flowfield are suggested as a generalization of Messiter's expansion to oscillating flow case

$$\frac{\bar{u}(\bar{x},\bar{y},\bar{z},\bar{t})}{\bar{U}\cos\alpha} = 1+\epsilon\tan^2\alpha\, u(x,y,z)$$

$$+\lambda_0 e^{ikt}U_1(x,y,z) + \cdots \tag{11a}$$

$$\frac{\bar{v}(\bar{x},\bar{y},\bar{z},\bar{t})}{\bar{U}\sin\alpha} = \epsilon v(x,y,z) + \lambda_0 e^{ikt}\{V_0(x,y,z) + \epsilon[V_1(x,y,z) - \tan\alpha\, u(x,y,z)]\} \quad (11b)$$

$$\frac{\bar{w}(\bar{x},\bar{y},\bar{z},\bar{t})}{\bar{U}\sin\alpha} = \epsilon^{1/2} w(x,y,z) + \lambda_0\epsilon^{1/2} e^{ikt} W_1(x,y,z) + \cdots \quad (11c)$$

$$\frac{\bar{p}(\bar{x},\bar{y},\bar{z},\bar{t}) - \bar{p}_\infty}{\bar{\rho}_\infty \bar{U}^2\sin^2\alpha} = 1 + \epsilon p(x,y,z) + \lambda_0 e^{ikt} P_1(x,y,z) + \cdots \quad (11d)$$

$$\frac{\bar{\rho}_\infty}{\bar{\rho}(\bar{x},\bar{y},\bar{z},\bar{t})} = \epsilon - \epsilon^2\rho(x,y,z) - \lambda_0\epsilon e^{ikt} R_1(x,y,z) + \cdots \quad (11e)$$

$$\bar{y}_s(\bar{x},\bar{z},\bar{t}) = \bar{y}_b(\bar{x},\bar{z},\bar{t}) + \bar{\ell}\epsilon\tan\alpha[y_s(x,z) + \lambda_0 e^{ikt} Y_1(x,z)] + \cdots \quad (11f)$$

where

$$k = \omega\bar{\ell}/(\bar{U}\cos\alpha) \quad (12)$$

is the reduced frequency. The basic assumption in the above representations is that they are uniformly valid throughout the region of interest between the wing surface and the shock wave. A nonuniformity for $\alpha \to \pi/2$ is expected in the representation for the function $\bar{u}$. But such case corresponds to the shock wave detached from the wing apex and is excluded from the present discussions. The dependence on α shown in the representations above and the appearance of the function u in Eq. (11b) are not essential for deriving the approximate equations but is a convenience that leads to an especially simple form of these equations. We substitute Eqs. (11) into the exact differential equations (4), the boundary condition (6), and the shock wave relations (7) and retain the lowest order terms in ϵ and λ_0. The function V_0 is found to be

$$V_0 = \cot\alpha\,[ik(h-x) - 1] \quad (13)$$

The other perturbation flow quantities are then to be determined from the following sets of equations.

System I

$$v_y + w_z = 0 \quad (14a)$$

$$u_x + v\,u_y + w\,u_z = 0 \quad (14b)$$

$$v_x + v\,v_y + w\,v_z + p_y = 0 \quad (14c)$$

$$w_x + v\,w_y + w\,w_z = 0 \quad (14d)$$

$$\left(\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)(p - \gamma\rho) = 0 \quad (14e)$$

At $y=0,\ v=0,\ 0<x<1,\ -\Omega x<z<\Omega x$ (15)

At $y = y_s(x,z)$,

$$u = -y_{sx} \quad (16a)$$

$$v = y_{sx} - y_{sz}^2 - 1 \quad (16b)$$

$$w = -y_{sz} \quad (16c)$$

$$p = 2y_{sx} - y_{sz}^2 - 1 \quad (16d)$$

$$\rho = a(2y_{sx} - y_{sz}^2) \quad (16e)$$

At $x=0,\ y_s=0$ (16f)

System II

$$R_{1x} + vR_{1y} + wR_{1z} + ikR_1 + U_{1x} + V_{1y} + W_{1z} + g\rho_y = 0 \quad (17a)$$

$$U_{1x} + vU_{1y} + wU_{1z} + ikU_1 + \tan^2\alpha\, gu_y = 0 \quad (17b)$$

$$P_{1y} + dV_0/dx + ikV_0 = 0 \quad (17c)$$

$$W_{1x} + vW_{1y} + wW_{1z} + ikW_1 + P_{1z} + w_xU_1 + w_yV_1 + w_zW_1 = 0 \quad (17d)$$

$$\left(\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + ik\right)(P_1 - \gamma R_1) = 0 \quad (17e)$$

At $y=0,\ V_1=0,\ 0<x<1,\ -\Omega x<z\Omega x$ (18)

At $y = y_s(x,z)$,

$$U_1 = \tan\alpha \quad (19a)$$

$$V_1 = \tan\alpha\, y_{sx} - 2y_{sz}Y_{1z} + [ik - v_y(x,y_s,z)]Y_1 + Y_{1x} + \cot\alpha(\bar{N} - y_{sz}^2)[(ikh - x) - 1] \quad (19b)$$

$$W_1 = -\{w_y(x,y_s,z)Y_1 + Y_{1z} + \cot\alpha\, y_{sz}[ik(h-x) - 1]\} \quad (19c)$$

$$P_1 = 2\cot\alpha[ik(h-x) - 1] \quad (19d)$$

$$R_1 = 2a\cot\alpha[ik(h-x) - 1] \quad (19e)$$

At $x=0,\quad Y_1=0$ (19f)

where

$$a = \frac{2}{2+N},\quad \bar{N} = \frac{2-N}{2+N},\quad N \equiv (\gamma-1)M_\infty^2\sin^2\alpha \quad (20)$$

The solution (13) for V_0 simply says that in a first approximation for the unsteady motion, the fluid in the thin shock layer moves with the same y-component of velocity as the wing. The largest change in pressure across the shock layer then results from the longitudinal acceleration of this fluid, as is seen from the y-momentum equation (17c).

System (I) is in exactly the same form as the problem formulated and studied by Messiter (Eqs. (1.19)-(1.21) of Ref. 11), for the steady flow over flat delta wings. Its solution gives a conical flow, and in particular the shock height is given in the following form

$$y_s = xy_s^*(\tilde{z}),\qquad \tilde{z} = z/x$$

Messiter has reduced the problem to that of a functional-differential equation (33) for $\tilde{z}(\tilde{w})$ (the same as $z(w)$ in Ref. 11) which he solved numerically only for values of Ω up to about 0.5, because the accuracy of the procedure used appeared to become poorer for increasing Ω. Messiter also obtained the first two terms in the series solution for the function $\tilde{z}$ in powers of Ω and used it to derive an analytical series solution for $F(\Omega)$ (Eq. (3.32) of Ref. 11), which is required in calculating the steady normal force. In deriving the analytical series solution for $F(\Omega)$ involving some complicated double integrals, further expansions in Ω and truncations of the integrands were carried out. However, a careful analysis shows

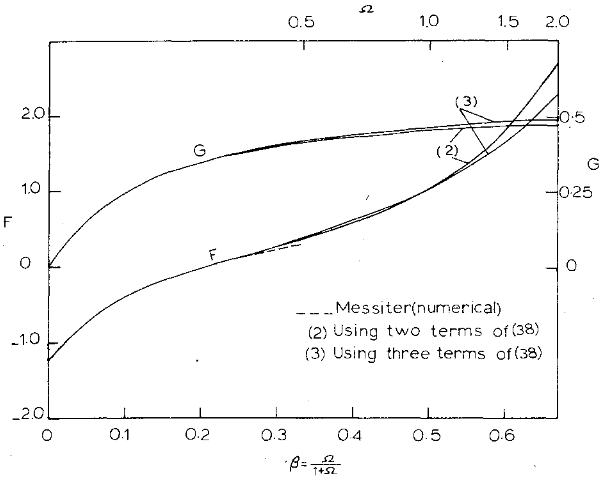

Fig. 2 The functions F and G.

that these further expansions are not uniformly valid over the whole range of integration except when $\Omega = 0$. This explains why the analytical series solution for $F(\Omega)$ is useless except for extremely small Ω, say $\Omega < 0.02$ as pointed out by Messiter. Therefore to obtain a valid approximation for $F(\Omega)$, the series solution for $\bar{z}$ should be used for direct numerical quadratures with no further expansions and truncations. In this way almost identical agreement (see Fig. 2) with Messiter's numerical solutions is obtained using either the first two terms or the first three terms in the series solution for $\bar{z}$. The three-term solution is expected to give good approximation for the whole range of $0 \le \Omega \le 2$. More details of analyses are given in Appendix A.

System (II) is linear and allows a solution for P_I in closed form. Thus from Eqs. (17c) and (19d) we obtain

$$P_I(x,y,z) = \cot\alpha\ [y - y_s(x,z)]\ [k^2(h-x) + 2\mathrm{i}k] + 2\cot\alpha\ [\mathrm{i}k(h-x) - 1] \tag{21}$$

Using Eqs. (13) and (21) in System (II), the latter simplifies to

$$U_{Ix} + V_{Iy} + W_{Iz} = \cot\alpha/\gamma\{k^2(h-x) + 2\mathrm{i}k]\ [2 + y_{sx} - v + w y_{sz} - \mathrm{i}k(y - y_s)] + k^2(y - y_s)\} \tag{22a}$$

$$U_{Ix} + vU_{Iy} + wU_{Iz} + \mathrm{i}k\ U_I = 0 \tag{22b}$$

$$W_{Ix} + vW_{Iy} + wW_{Iz} + \mathrm{i}kW_I + w_xU_I + w_yV_I + w_zW_I = = \cot\alpha\ y_{sz}[k^2(h-x) + 2\mathrm{i}k] \tag{22c}$$

$$\text{At } y=0,\quad V_I = 0,\quad 0 < x < 1,\quad -\Omega x < z < \Omega x \tag{23}$$

At $y = y_s(x,z)$

$$U_I = \tan\alpha \tag{24a}$$

$$V_I = \tan\alpha\ y_{sx} - 2y_{sz}Y_{Iz} + [\mathrm{i}k - v_y(x,y_s,z)]\,Y_I + Y_{Ix} + \cot\alpha[\mathrm{i}k(h-x) - 1]\,(2a - 1 - y_{sz}^2) \tag{24b}$$

$$W_I = -\{w_y(x,y_s,z)Y_I + Y_{Iz} + \cot\alpha[\mathrm{i}k(h-x) - 1]y_{sz}\} \tag{24c}$$

$$\text{At } x=0,\quad Y_I = 0 \tag{24d}$$

Equation (24d) holds because the shock is assumed attached to the wing apex.

It can easily be shown that the system of Eqs. (22-24) allow quasi-conical solutions of the form

$$U_I(x,y,z) = xU^*(y/x,\ z/x) + h\tilde{U}(y/x,\ z/x),\ \text{etc.} \tag{25}$$

However, for the purpose of studying the aerodynamic performance of the wing the functions U_I, V_I, and W_I are not needed unless one should want to calculate higher approximations; the system of Eqs. (22-24) will not be discussed further.

III. The Aerodynamic Derivatives

With the simple closed-form solution Eq. (21) for the unsteady pressure $P_I(x,y,z)$, one can derive closed-form formulas for the aerodynamic derivatives. First, the pitching moment coefficient C_m is given by

$$C_m = \frac{\bar{M}}{\tfrac{1}{2}\bar{\rho}_\infty \bar{U}^2\ \ell^2 \bar{b}} = 2\int_0^{\ell}\int_0^{b\bar{x}} [\bar{p}(\bar{x},\bar{y}_b,\bar{z},\bar{t}) - \bar{p}_0]\ (\bar{x} - h\ell)\ \mathrm{d}\bar{z}\mathrm{d}\bar{x} \tag{26}$$

where $\bar{M}$ is the pitching moment, and $\bar{p}_0$ the value of $\bar{p}$ when the wing is stationary in the plane $\bar{y} = 0$. Introducing

$$\bar{z} = z/x$$

and let $P_I(x,o,z) = \tilde{P}_I(x,o,\bar{z})$, we get

$$C_m = \frac{4}{b}\ \epsilon^{1/2}\lambda_0 \sin^2\alpha\ \tan\alpha\ e^{\mathrm{i}kt}\int_0^1\int_0^{\Omega} x\ (x-h)\tilde{P}_I(x,o,\bar{z})\,\mathrm{d}\bar{z}\ \mathrm{d}x \tag{27}$$

Since

$$C_m = \lambda_0 e^{\mathrm{i}kt}[m_\theta + \mathrm{i}k\ m_{\dot\theta}] \tag{28}$$

where $-m_\theta$ and $-m_{\dot\theta}$ represents, respectively, the in-phase and the out-of-phase components of the aerodynamic derivatives. Thus we get the following formulas for $-m_\theta$, and $-m_{\dot\theta}$

$$-m_\theta = 2\sin 2\alpha\left[\frac{2}{3} - h + k^2G(\Omega)\ \left(\frac{1}{2}h - \frac{1}{3}h^2 - \frac{1}{5}\right)\right] \tag{29}$$

$$-m_{\dot\theta} = 2\sin 2\alpha\left[\left(\frac{1}{2} - \frac{2}{3}h\right)G(\Omega) + h^2 - \frac{4}{3}h + \frac{1}{2}\right] \tag{30}$$

where

$$G(\Omega) = \frac{1}{\Omega}\int_0^{\Omega} y_s^*(\bar{z})\ \mathrm{d}\bar{z} \tag{31}$$

is the average shock height. The function $y_s^*(\bar{z})$ (the same as the function $y_s(z;\Omega)$ in Ref. 11) has to be determined by solving System (I). Indeed, according to Messiter (Eq. (3.29) of Ref. 11)

$$y_s^*(\bar{w}) = \int_{\bar{z}(\bar{w})}^{\bar{w}} \frac{(w_I - \bar{z}(\bar{w}))}{[w_I - \bar{z}(w_I)]^2}\ \bar{z}'(w_I)\ \mathrm{d}w_I \tag{32}$$

where $\bar{z}(\bar{w})$ satisfies the following functional-differential equation[11]

$$\frac{\bar{z}'(\bar{w})\bar{z}'[\bar{z}(\bar{w})]}{\{\bar{z}[\bar{z}(\bar{w})] - \bar{z}(\bar{w})\}^2} = \frac{1}{[\bar{z}(\bar{w}) - \bar{w}]^2} - 1 \tag{33a}$$

$$\bar{z}(0) = 0 \tag{33b}$$

$$\bar{z}(1+\Omega) = \Omega \tag{33c}$$

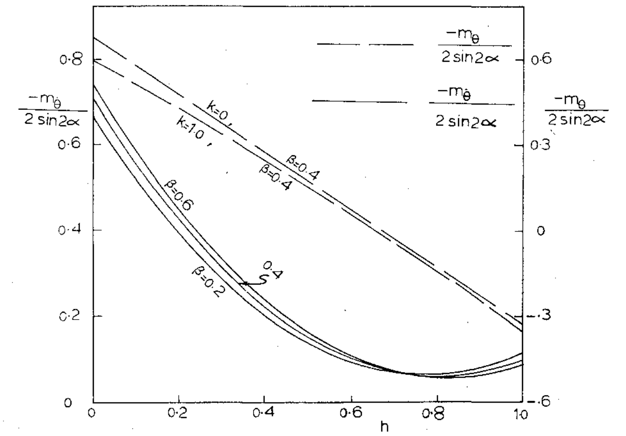

Fig. 3 Aerodynamic derivatives versus pivot position. $\beta = \Omega/(1+\Omega)$

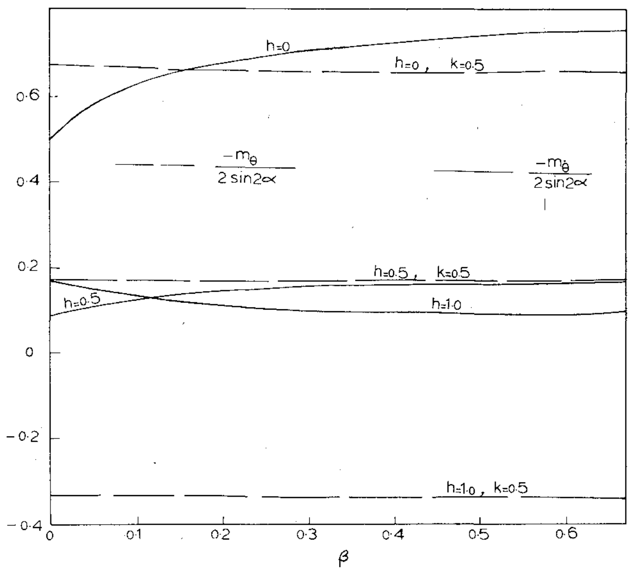

Fig. 4 Aerodynamic derivatives versus the parameter β.

and, due to flow symmetry, $\bar{z}(\bar{w})$ is an odd function. An accurate perturbation solution to Eq. (33) will be given in Appendix A. In particular, the function $G(\Omega)$, which is required for calculating the aerodynamic derivatives, is obtained and presented in Fig. 2.

Using the result for $G(\Omega)$, we now discuss the aerodynamic derivatives of a pitching delta wing with detached shock wave as follows. Although according to Eq. (28) both the in-phase component $-m_\theta$ and the out-of-phase component $-m_{\dot\theta}$ in general are infinite series in k^2, it turns out in the present case that $-m_{\dot\theta}$ is independent of the frequency k, whereas $-m_\theta$ includes only terms up to k^2. Figure 3 shows that $-m_\theta$ only slightly varies with k for values of k up to 1.0.

Equations (29) and (30) show that the aerodynamic derivatives $-m_\theta/2 \sin 2\alpha$ and $-m_{\dot\theta}/2 \sin 2\alpha$ do not depend on the flight parameters M_∞, α, γ and the wing aspect ratio b separately, but rather depend only on their combination as represented by $G(\Omega)$, showing the importance of the similarity parameter Ω in analyzing the aerodynamic stability of a delta wing. Moreover, Eq. (29) shows that the principal part of $-m_\theta/2 \sin 2\alpha$, i.e. when $k=0$, does not depend on any flight parameter (M_∞, α, γ, b) and that the aerodynamic center is at $h = ⅔$ in accord with Messiter's result.

As seen from Figs. 3 and 4, the effect of G, i.e. of the aspect ratio, is negligible on $-m_\theta$ but moderate on $-m_{\dot\theta}$. Similar conclusions were reached by Liu and Hui[12] for pitching delta wings with attached shock waves.

Finally, the out-of-phase aerodynamic derivative $-m_{\dot\theta}$ may be rewritten

$$\frac{-m_{\dot\theta}}{2 \sin 2\alpha} = [h - \frac{1}{3}(G+2)]^2 + \frac{1}{18}(1-G)(1+2G) \tag{34}$$

But Fig. 2 shows that $G(\Omega) < 1$ for $0 \le \Omega \le 2$, hence $-m_{\dot\theta} > 0$. We therefore conclude that within the thick shock-layer approximation, the pitching motion of slender flat delta wings with detached shock waves in hypersonic flight is always stable aerodynamically. In this context it is noted that in the case of pitching delta wings with attached shock wave[12] and in the case of pitching wedges,[3] the motion may become aerodynamically unstable if the flight Mach number is low enough.

In conclusion, aerodynamic stability of slender delta wings with detached shock waves performing small amplitude pitching motion in hypersonic flight is studied within thin shock-layer approximation. Closed-form formulas are derived for the aerodynamic derivatives and it is shown that the pitching motion is always stable aerodynamically.

Appendix A: Solution of the Functional-Differential Equation (33)

It was mentioned in Sec. II that Messiter obtained the first two terms in the series solution to Eq. (33) for $\bar{z}(\bar{w})$ in Ω. It was also mentioned that in using this two-term solution of $\bar{z}$ to derive an analytical series solution for $F(\Omega)$ (Eq. (3.32) of Ref. 11) further series expansion of certain functions and truncation are involved that are not valid, thus rendering the analytic series solution for $F(\Omega)$ useless. In this appendix the first three terms in the series solution to Eq. (33) will be obtained and used to derive accurate approximations for the functions F and G, the latter being required for calculating the stability derivatives.

For convenience, we introduce the following transformation

$$z^*(w^*) = \bar{z}(\bar{w})/\Omega, \quad w^* = \bar{w}/(1+\Omega), \quad \beta = \Omega/(1+\Omega) \tag{35}$$

then Eq. (33) becomes

$$z^{*\prime}(w^*) z^{*\prime}[\beta z^*(w^*)] (\beta z^*(w^*) - w^*)^2 =$$

$$\{z^*[\beta z^*(w^*)] - z^*(w^*)\}^2$$

$$\left\{1 - \left[\frac{(\beta z^*(w^*) - w^*)}{(1-\beta)}\right]^2\right\} \tag{36a}$$

$$z^*(0) = 0 \tag{36b}$$

$$z^*(1) = 1 \tag{36c}$$

The parameter β takes values in the range

$$0 \le \beta \le ⅔, \qquad \text{for} \quad 0 \le \Omega \le 2 \tag{37}$$

Assume a perturbation solution in the form

$$z^*(w^*;\beta) = z_0(w^*) + \beta z_1(w^*) + \beta^2 z_2(w^*) + \cdots \tag{38}$$

Substituting Eqs. (38) into (36) and expanding and equating like powers of β, we get the following three sets of problems for the successive determination of the odd functions $z_0(w^*)$, $z_1(w^*)$, and $z_2(w^*)$

$$z_0'(0) \frac{z_0'(w^*)}{z_0^2(w^*)} = \frac{1 - w^{*2}}{w^{*2}} \tag{39a}$$

$$z_0(0) = 0 \tag{39b}$$

$$z_0(1)=1 \tag{39c}$$

$$z_1'(w^*)-\frac{(1-w^{*2})}{w^{*2}}z_0(w^*)z_1(w^*)=\frac{2z_0(w^*)z_0'(w^*)}{w^*}$$

$$-\frac{1}{2}z_0'(w^*)z_1'(0)-z_0^2(w^*)[1-\frac{z_0(w^*)}{w^*}]$$

$$-2[\frac{z_0(w^*)}{w^*}]^2(1-w^{*2}) \tag{40a}$$

$$z_1(0)=0 \tag{40b}$$

$$z_1(1)=0 \tag{40c}$$

and

$$z_2'(w^*)-\frac{(1-w^{*2})}{w^{*2}}z_0(w^*)z_2(w^*)$$

$$=\frac{z_0(w^*)}{w^*}\Big\{(\pi-2)z_0'(w^*)+2z_1'w^*)\Big\}$$

$$-2z_0'(w^*)\Big\{\frac{1}{2}\Big[\frac{z_0(w^*)}{w^*}\Big]^2$$

$$-\frac{z_1(w^*)}{w^*}\Big\}-½z_0'(w^*)\{z_2'(0)$$

$$-6z_0^2(w^*)\}-\Big(\frac{\pi}{2}-1\Big)z_1'(w^*)$$

$$-½\Big[\frac{z_0(w^*)}{w^*}\Big]^2\{z_0^2(w^*)-2w^*z_1(w^*)$$

$$-4w^*z_0(w^*)+3w^{*2}\}$$

$$+2z_0(w^*)\{2z_0(w^*)-z_1(w^*)\}\Big\{1-\frac{z_0(w^*)}{w^*}\Big\}$$

$$+(1-w^{*2})\Big\{\frac{1}{2}\Big[\frac{2z_0(w^*)}{w^*}-\frac{z_1(w^*)}{w^*}\Big]^2$$

$$-\frac{z_0(w^*)}{w^*}\Big[2\frac{z_1(w^*)}{w^*}+(\pi-2)\frac{z_0(w^*)}{w^*}\Big]\Big\} \tag{41a}$$

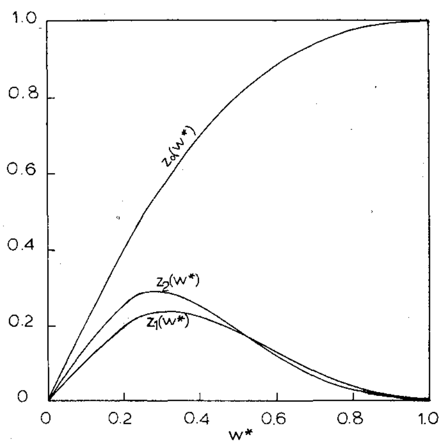

Fig. 5 The universal functions z_0, z_1, z_2.

$$z_2(0)=0 \tag{41b}$$

$$z_2(1)=0 \tag{41c}$$

Solutions for z_0 and z_1 are given, respectively, by

$$z_0(w^*)=2w^*/(1+w^{*2}) \tag{42}$$

$$z_1(w^*)=\frac{w^*}{(1+w^{*2})^2}\{(\pi+2)\ (1+w^{*2})$$

$$-8w^*\tan^{-1}w^*-4\} \tag{43}$$

System (41) is a boundary value problem that is solved numerically by the method of iteration and the result is plotted in Fig. 5 together with $z_0(w^*)$ and $z_1(w^*)$, in particular we have

$$z_2'(0)=1.6165 \tag{44}$$

The first two-term solutions of Eq. (38) as given by Eqs. (42) and (43) can, of course, also be obtained from Messiter's two-term solution via the transformation Eq. (35). But the advantage of using the new variables is that the functions $z_0(w^*)$, $z_1(w^*)$, and $z_2(w^*)$, etc., are independent of any parameter. Thus they are universal for all flat delta wings under all flight conditions.

We now use the series solution obtained above to calculate the functions G defined in Eq. (31) and F defined in Ref. 11. These are

$$G=\beta\int_0^1 z^{*\prime}(w)\,I_1(w,\beta)\ dw \tag{45a}$$

$$F=2\int_0^1\Big[z^{*\prime}(w)\Big\{\frac{2\beta wz^*(w)-w^2}{(1-\beta)^2}-1+2\beta I_1(w,\beta)\Big\}$$

$$+\Big\{1-\frac{(w-\beta z^*(w))^2}{(1-\beta)^2}\Big\}I_2(w,\beta)\Big]dw \tag{45b}$$

where

$$I_1(w,\beta)=\int_{\beta z^*(w)}^{w}K(w,\ w_1,\ \beta)\ dw_1 \tag{46a}$$

$$I_2(w,\beta)=\int_{\beta z^*(w)}^{w}\Big[\frac{w_1-\beta z^*(w)}{w-\beta z^*(w)}\Big]^2K(w,w_1,\beta)\ dw_1 \tag{46b}$$

$$K(w,\ w_1,\beta)=\frac{[w_1-\beta z^*(w)]z^{*\prime}(w_1)}{[w_1-\beta z^*(w_1)]^2} \tag{47}$$

Now if the first two terms of the series solution Eq. (38) for z^* are used in Eq. (46) and the integrand K is expanded as Taylor's series in β, one obtains

$$F=-1.228-8\beta\ \ell n\ \beta-17.17\beta+\cdots \tag{48}$$

which is, to the same order in β or in Ω, just Messiter's analytic series solution for F. But the expansion of the integrand $K(w,w_1,\beta)$ as Taylor's series in β is not uniformly valid over the whole range of integration. In particular, it is not valid near the lower limit $\beta z^*(w)$ because‡

$$\lim_{\substack{\beta\to 0\\ w_1\to\beta z^*(w)\\ w\ \text{fixed}}}\frac{\partial[w_1K(w,w_1,\beta)]}{\partial\beta}=\infty \tag{49}$$

‡The same is true if Ω is used instead of β.

Therefore, to obtain correct approximations for F and G, the series solution Eq. (38) must be used to evaluate the integrals I_1 and I_2 with no further expansion of the integrands in series in β. The numerical difficulties in evaluating I_1 and I_2 for small w are overcomed by introducing the following transformation

$$w_1 = [w - \beta z^*(w)] w_2 + \beta z^*(w) \tag{50}$$

Accordingly,

$$I_1(w, \beta) = \int_0^1 H(w, w_2, \beta)\, dw_2 \tag{51a}$$

$$I_2(w, \beta) = \int_0^1 H(w, w_2, \beta)\, w_2^2\, dw_2 \tag{51b}$$

$$H(w, w_2, \beta) = w_2 z^{*\prime}(w_1) \left[\frac{w - \beta z^*(w)}{w_1 - \beta z^*(w_1)} \right]^2 \tag{52}$$

is well-behaved throughout the whole range of integration.

Results obtained for the functions F and G using the first two terms and using the first three terms of Eq. (38) are presented in Fig. 2. It is seen that the results for F either by using two terms or three terms of Eq. (38) are almost identical with Messiter's numerical solution for Ω up to 0.5. The fact that (see Fig. 2) there is only a small difference between the two-term and the three-term results for both F and G indicates the rapid convergence and thus the high accuracy of the approximations. These approximations for F and G may be used for the whole range of $0 \leq \Omega \leq 2$.

References

[1] Appleton, J. P., "Aerodynamic Pitching Derivatives of a Wedge in Hypersonic Flow," *AIAA Journal,* Vol. 2, Nov. 1964, pp. 2034-2036.

[2] McIntosh, S. C., Jr., "Hypersonic Flow over an Oscillating Wedge," *AIAA Journal,* Vol. 3, March 1965, pp. 433-440.

[3] Hui, W. H., "Stability of Oscillating Wedges and Caret Wings in Hypersonic and Supersonic Flows," *AIAA Journal,* Vol. 7, Aug. 1969, pp. 1524-1530.

[4] Hui, W. H., "Interaction of a Strong Shock with Mach Waves in Unsteady Flow," *AIAA Journal,* Vol. 7, Aug. 1969, pp. 1605-1607.

[5] Kuiken, A. K., "Large-Amplitude Low-Frequency Oscillation of a Slender Wedge in Inviscid Hypersonic Flow," *AIAA Journal,* Vol. 7, Sept. 1969, pp. 1767-1774.

[6] Hui, W. H., "Large-Amplitude Slow Oscillation of Wedges in Inviscid Hypersonic and Supersonic Flows," *AIAA Journal,* Vol. 8, Aug. 1970, pp. 1530-1532.

[7] Orlik-Ruckemann, K. J., "Stability Derivatives of Sharp Wedges in Viscous Hypersonic Flow," *AIAA Journal,* Vol. 4, June 1966, pp. 1001-1007.

[8] Hui, W. H. and East, R. A., "Stability Derivatives of Sharp Wedges in Viscous Hypersonic Flow," *Aeronautical Quarterly,* Vol. 22, May, 1971, pp. 127-145.

[9] Hui, W. H., "Supersonic and Hypersonic Flow with Attached Shock Waves over Delta Wings," *Proceedings of the Royal Society of London, Series A,* Vol. 325, 1971, pp. 251-267.

[10] Hui, W. H., "Methods for Calculating Nonlinear Flow with Attached Shock Waves over Conical Wings," *AIAA Journal,* Vol. 11, Oct. 1973, pp. 1443-1445.

[11] Messiter, A. F., "Lift of Slender Delta Wings According to Newtonian Theory," *AIAA Journal,* Vol. 1, April 1963, pp. 794-802.

[12] Liu, D. D. and Hui, W. H., "Oscillating Delta Wings with Attached Shock Waves," to be published.

[13] Hida, K., "Thickness Effect on the Force of Slender Delta Wings in Hypersonic Flow," *AIAA Journal,* Vol. 3, March 1965, pp. 427-433.

[14] Squire, L. C., "Calculated Pressure Distributions and Shock Shapes on Thick Conical Wings at High Supersonic Speeds," *Aeronatical Quarterly,* Vol. 18, May 1967, pp. 185-206.